Basic Linear Algebra

with Applications to Data Analysis

Carl O. R. Lutz
carl.lutz@uni.lu
carlorlutz.github.io

Université du Luxembourg
September 3, 2026

Refresher I

— Basic Notions —

- Groups, Fields, Vector Spaces
- Subspaces, Spanning Sets, Bases
- Direct Sums, Linear Complements
- Linear Functions, Isomorphisms
- Rank, Kernel, Rank-Nullity Theorem

Groups, Fields & Vector Spaces

Definition (Group)

A group is a set G with a binary operation $\cdot : G \times G \to G$ s.t.

- Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity element: $\exists! e \in G \forall a \in G : a \cdot e = a = e \cdot a$
- Inverse element: $\forall a \in G \exists! a^{-1} \in G : a \cdot a^{-1} = e = a^{-1} \cdot a$

The group is abelian/commutative if

$$a + b = b + a \qquad \forall a, b \in G.$$

Groups, Fields & Vector Spaces

Definition (Field)

A field is a set $\mathbb{K}$ with binary operations $+, \cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ s.t.

- $(\mathbb{K}, +)$ is an abelian group with identity element 0;
- $\mathbb{K}_* := \mathbb{K} \setminus \{0\}$ is an abelian group with identity element 1;
- $\forall x, y, z \in \mathbb{K}$:

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Groups, Fields & Vector Spaces

Definition (Vector Space)

A vector space over a field $\mathbb{K}$ is a set V with a binary operation $+\colon V \times V \to V$ and a *scalar multiplication* $\cdot\colon \mathbb{K} \times V \to V$ s.t.

- $(V, +)$ is an abelian group;
- $+$ and $\cdot$ are compatible in the sense that for all $\lambda, \lambda_1, \lambda_2 \in \mathbb{K}$ and $v, w \in V$ holds
  - (i) $1 \cdot v = v$;
  - (ii) $(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v)$.
  - (iii) $(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v)$;
  - (iv) $\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2)$;

Subspaces & Spanning Sets

Definition

A linear subspace of a vector space V over $\mathbb{K}$ is a subset $U \subset V$ that is closed under addition and scalar multiplication:

- $0 \in U$;
- if $u_1, u_2 \in U$ then $u_1 + u_2 \in U$;
- if $u \in U$ then $\lambda \cdot u \in U$ for all $\lambda \in \mathbb{K}$.

Definition

The span of a set $\mathcal{S} \subset V$ is

$$\operatorname{span}(\mathcal{S}) := \left\{ \sum_{i=1}^{N} \lambda_i \cdot s_i \mid N \in \mathbb{N}, s_i \in S, \lambda_i \in \mathbb{K} \right\}.$$

If $\operatorname{span}(\mathcal{S}) = V$, then $\mathcal{S}$ is a spanning set of V.

Bases of Vector Spaces

Definition

A set $\mathcal{B} \subset V \setminus \{0\}$ is called linear independent if for all $N \in \mathbb{N}$ and $b_1, \ldots, b_N \in \mathcal{B}$ holds:

$$\lambda_1 b_1 + \ldots + \lambda_N b_N = 0 \quad \Longleftrightarrow \quad \lambda_1 = \ldots = \lambda_N = 0.$$

If $\mathcal{B}$ is also a *spanning set* of V, then $\mathcal{B}$ is called a basis of V. The dimension of V is the cardinality $\dim_{\mathbb{K}}(V) = |\mathcal{B}|$. In particular, V is finite dimensional if $\dim_{\mathbb{K}}(V) < \infty$.

New from Old: Union ($\cup$) & Intersection ($\cap$)

Proposition

Let V_1, V_2 be subspaces of V. Then

- $V_1 \cap V_2$ is *always* a subspace of V;
- $V_1 \cup V_2$ is a linear subspace of V *if and only if* $V_1 \subset V_2$ or $V_2 \subset V_1$.

Proposition

Let $\mathcal{S} \subset V$. The span of $\mathcal{S}$ is the smallest linear subspace of V containing $\mathcal{S}$ in the sense that

$$\operatorname{span}(\mathcal{S}) = \bigcap_{\substack{\mathcal{S} \subset U \subset V \\ \text{subspace}}} U.$$

New from Old: Sum (+) & Direct Sum ($\oplus$)

Definition (Direct Sum)

Let V, W be vector spaces over $\mathbb{K}$. The direct sum $V \oplus W$ is the set $V \times W$ endowed with the

- *addition*: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$;
- *scalar mult.*: $\lambda \cdot (u, w) = (\lambda v, \lambda w)$.

Definition (Sum & Linear Complement)

Let V_1, V_2 be linear subspaces of V.

- The sum of V_1 and V_2 is the subspace $V_1 + V_2 \coloneqq \operatorname{span}(V_1 \cup V_2)$.
- V_1 and V_2 are linear complements (in V) if $V_1 \oplus V_2$ is isomorphic to V.

Linear Functions

Definition

Let V, W be vector spaces over $\mathbb{K}$.

- A linear function is a function $f \colon V \to W$ such that

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$$

  for all $\lambda_1, \lambda_2 \in \mathbb{K}$ and $v_1, v_2 \in V$.

- A linear function is called isomorphism if it admits a linear inverse $f^{-1} \colon W \to V$, *i.e.*, f^{-1} is linear and $f^{-1} \circ f = \mathrm{id}_V$.
  In this case we call V and W *(linear) isomorphic* and write $V \simeq W$.

Proposition

Let $\mathcal{B} \subset V$ be a basis. Every linear map $f: V \to W$ is determined by its action on $\mathcal{B}$, *i.e.*, if $w_b \in W$ for $b \in \mathcal{B}$, then

$$f(b) := w_b$$

defines f completely.

Proposition

The set $\mathrm{Lin}(V, W)$ of linear functions from V to W is a vector space. If $\dim(V), \dim(W) < \infty$, then $\dim(\mathrm{Lin}(V, W)) = \dim(V) \cdot \dim(W)$.

Proposition

The set $\mathrm{GL}(V)$ of isomorphisms $V \to V$ is a group.

Basics of Matrices

Definition

An (m, n)-matrix over a field $\mathbb{K}$ is a map $A\colon \{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{K}$.
We usually write

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} = (a_{ij})_{mn} = (a_{ij})$$

For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ is $\begin{pmatrix} a_{i1} & \ldots & a_{in} \end{pmatrix}$ the ith row and $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ the jth column of A.
The set of all (m, n)-matrices of $\mathbb{K}$ is denoted by $\mathbb{K}^{m,n}$.

Basics of Matrices

Definition

- The matrix product $\cdot\colon \mathbb{K}^{m,n} \times \mathbb{K}^{n,p} \longrightarrow \mathbb{K}^{m,p}$ is

$$(a_{ij})_{mn} \cdot (b_{ij})_{np} := \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{mp}$$

- The *transpose* of a matrix $A = (a_{ij})_{mn} \in \mathbb{K}^{m,n}$ is $A^T = (\tilde{a}_{ij})_{nm} \in \mathbb{K}^{n,m}$ with $\tilde{a}_{ij} = a_{ji}$.

Matrices & Linear Functions

Proposition

- Each $A \in \mathbb{K}^{m,n}$ defines a linear function $L_A \colon \mathbb{K}^n \to \mathbb{K}^m$ via

$$L_A(x) := A \cdot v, \qquad v \in \mathbb{K}^n.$$

- For all $A \in \mathbb{K}^{m,n}$ and $B \in \mathbb{K}^{n,p}$ holds $L_A \circ L_B = L_{A \cdot B}$.

Matrices & Linear Functions

Proposition

Let $\mathcal{B} \subset V$, $\mathcal{B}' \subset W$ be bases with $\dim V = n$ and $\dim W = m$.
Then for $f \in \mathrm{Lin}(V, W)$ and all $j \in \{0, \ldots, n\}$ there are $a_{1j}, \ldots, a_{mj} \in \mathbb{K}$ such that

$$f(b_j) = a_{1j}b'_1 + \ldots + a_{mj}b'_m.$$

The matrix representation of f is the matrix $[f]_{\mathcal{B},\mathcal{B}'} := (a_{ij})_{ij}$.
It satisfies that for $v = \lambda_1 b_1 + \ldots + \lambda_n b_n \in \mathbb{K}^n$ and $\mu := L_{[f]_{\mathcal{B},\mathcal{B}'}}(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^m$ holds

$$f(v) = \mu_1 b'_1 + \ldots + \mu_m b'_m.$$

Rank-Nullity of Linear Functions

Definition

Let $f: V \to W$ be a linear function and $A \in \mathbb{K}^{m,n}$.

- The kernel of f is
$$\ker(f) := \{v \in V \mid f(v) = 0\}.$$
If $\dim(V) < \infty$, then the nullity of f is $\dim\ker(f)$.
- The image of f is
$$\mathrm{im}(f) := \{f(v) \in W \mid v \in V\}.$$
If $\dim(V) < \infty$, then the rank of f is $\dim\mathrm{im}(f)$.
- We define $\ker(A) := \ker(L_A)$ and $\mathrm{im}(A) := \mathrm{im}(L_A)$.

Rank-Nullity of Linear Functions

Theorem (Rank-Nullity)

Let $f : V \to W$ be a linear function and $A \in \mathbb{K}^{m,n}$. Then

- $V \simeq \ker(f) \oplus \operatorname{im}(f)$
- *If* $\dim(V) < \infty$, *then* $\dim(V) = \dim\ker(f) + \dim\operatorname{im}(f)$
- $n = \dim\ker(L_A) + \dim\operatorname{im}(L_A)$

The Determinant

Definition

The determinant $\det\colon \mathbb{K}^{n,n} \to \mathbb{K}$ is

$$\det A := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n(\sigma n)}.$$

Proposition

Some of the properties of the determinant are:

- $\forall A, B \in \mathbb{K}^{n,n}\colon\ \det(A \cdot B) = \det(A)\det(B)$
- $\det(A) = 0 \Leftrightarrow A \in \mathrm{GL}(\mathbb{K}; n)$
- $\det$ is multi-linear, but <u>not</u> linear!

Determinant — Adjugate Formula

Definition

- Let $A \in \mathbb{K}^{n,n}$. The ij-minor matrix $A^{ij} \in \mathbb{K}^{n-1,n-1}$ is obtained by deleting the ith row and jth column of A.
- The adjugate $\mathrm{adj}(A) \in \mathbb{K}^{n,n}$ of $A \in \mathbb{K}^{n,n}$ is given by

$$\mathrm{adj}(A)_{ij} := (-1)^{i+j} \det(M^{ji}).$$

Proposition

For all $A \in \mathbb{K}^{n,n}$ holds

$$A \,\mathrm{adj}(A) = \mathrm{adj}(A) A = \det(A) I_n.$$

In particular, for $A \in \mathrm{GL}(\mathbb{K}; n)$ we have

$$A^{-1} = \frac{1}{\det(A)} \,\mathrm{adj}(A).$$

Determinant — Isomorphisms

Proposition

A map $f \in \mathrm{Lin}(V, W)$ between finite dimensional vector spaces is an isomorphism *if and only if* there exists a representation as a square matrix $[f]_{\mathcal{B}_V,\mathcal{B}_W} \in \mathbb{K}^{n,n}$ with $\det([f]_{\mathcal{B}_V,\mathcal{B}_W}) \neq 0$.
Moreover, the value of $\det([f]_{\mathcal{B}_V,\mathcal{B}_W})$ is independent of the choice of bases $\mathcal{B}_V$ and $\mathcal{B}_W$.

Refresher II

— Eigenvalue Problems —

- Eigenvalues and Eigenvectors
- Characteristic Polynomial
- Criteria for Diagonalizability
- Jordan Decomposition

Eigenvalues & Eigenvectors

Definition

- An eigenvalue-eigenvector pair of $f \in \mathrm{Lin}(V, V)$ is are $\lambda \in \mathbb{K}$ and $v \in V \setminus \{0\}$ s.t.
$$f(v) = \lambda v.$$
- The eigenpairs of $A \in \mathbb{K}^{n,n}$ are the eigenpairs of $L_A \in \mathrm{Lin}(\mathbb{K}^n, \mathbb{K}^n)$.

Proposition

The eigenspace $\mathrm{Eig}(f; \lambda) \subset V$ of f for $\lambda \in \mathbb{K}$ is the vector space of eigenvectors of f for λ. In particular,

$$\mathrm{Eig}(f; \lambda) = \ker(\lambda \,\mathrm{id}_V - f).$$

Characteristic Polynomial

Definition

Let $A \in \mathbb{K}^{n,n}$. The characteristic polynomial of A is

$$\chi_A(t) := \det(tI_n - A) \in \mathbb{K}[t].$$

Proposition

- $\chi_A(t)$ is monic of degree n. In particular,

$$\chi_A(t) = t^n - \operatorname{tr}(A) + \cdots + (-1)^n \det(A).$$

- $\lambda \in \mathbb{K}$ is an eigenvalue of A *if and only if* λ is a root of $\chi_A(t)$.

Criteria for Diagonalizability

Definition

A square matrix $A \in \mathbb{K}^{n,n}$ is called diagonalizable if there exists $S \in \mathrm{GL}(\mathbb{K}; n)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Proposition

Let $A \in \mathbb{K}^{n,n}$.

- A is diagonalizable *if and only if* $\mathbb{K}^n$ possesses a basis of eigenvectors of A.
- If $\chi_A(t)$ has n distinct roots, then A is diagonalizable.

Minimal Polynomial

Definition

Let $A \in \mathbb{K}^{n,n}$. The minimal polynomial $\mu_A(t) \in \mathbb{K}[t]$ of A is the monic polynomial of smallest degree that satisfies $\mu_A(A) = 0$.

Proposition

Let $A \in \mathbb{K}^{n,n}$.

- The characteristic polynomial $\chi_A(t)$ is a multiple of $\mu_A(t)$, *i.e.*, $\exists q(t) \in \mathbb{K}[t]$ with $\chi_A(t) = q(t)\mu_A(t)$.
- A is diagonalizable *if and only if* $\mu_A(t)$ splits over $\mathbb{K}$ and has distinct roots, *i.e.*, $\exists m < n, \lambda_i \in \mathbb{K}$ with $\lambda_i \neq \lambda_j$ s.t. $\mu_A(t) = \prod_{i=1}^{m}(t - \lambda_i)$.

Corollary

Every projector $P \in \mathrm{Lin}(V, V)$, $\dim(V) < \infty$, is diagonalizable with eigenvalues in $\{0, 1\}$.

Jordan Decomposition

Definition

The $n \times n$ Jordan block for $\lambda \in \mathbb{K}$ is $J_{\lambda,n} := \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in \mathbb{K}^{n,n}$.

Theorem (Jordan Decomposition)

Let $A \in \mathbb{K}^{n,n}$ and assume that $\chi_A(t)$ splits over $\mathbb{K}$. Then there is a $k \in \mathbb{N}$, $(\lambda_i, n_i) \in \mathbb{C} \times \mathbb{N}$ and $S \in \mathrm{GL}(\mathbb{K}; n)$ s.t.

$$S^{-1}AS = \begin{pmatrix} J_{\lambda_1,n_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_k,\lambda_k} \end{pmatrix}.$$

Jordan decomposition — Some Corollaries

Theorem (Fundamental Theorem of Algebra)

Every polynomial with coefficients in $\mathbb{C}$ splits, i.e., for every $p(t) \in \mathbb{C}[t]$ there exist $c, \lambda_1, \cdot, \lambda_n \in \mathbb{C}$ with $p(t) = c(t - \lambda_1) \cdot \ldots \cdot (t - \lambda_n)$.

Corollaries

- Every $A \in \mathbb{C}^{n,n}$ possesses a Jordan decomposition.
- For $A \in \mathbb{C}^{n,n}$ holds $\det(A) = \prod_{i=1}^{k} \lambda_i^{n_i}$ and $\operatorname{tr}(A) = \sum_{i=1}^{k} n_i \lambda_i$.

Jordan Decomposition — Computation

Let $A \in \mathbb{K}^{n,n}$.

(i) Factorize $\chi_A(t)$: find $m, n_i \in \mathbb{N}$ and $\lambda_i \in \mathbb{K}$ s.t. $\chi_A(t) = \prod_{i=1}^{m}(t-\lambda_i)^{n_i}$.

(ii) $\forall i$: find $k_i \in \mathbb{N}$ s.t $(I_n - \lambda_i A)^{k_i - 1} \neq 0$ and $(I_n - \lambda_i A)^{k_i} = 0$.

(iii) $\forall i$: Compute a linear complement V_i of

$$\ker(I_n - \lambda_i A)^{k_i - 1} \subset \ker(I_n - \lambda_i A)^{k_i}.$$

(iv) $\forall i$: Compute a basis $v_1^{(i)}, \dots, v_{l_i}^{(i)}$ of V_i.

(v) Construct the $S \in \mathrm{GL}(\mathbb{K}; n)$ with column vectors

$$v, \quad (\lambda_i I_n - A)v, \quad \dots, (\lambda_i I_n - A)^{k-1} v.$$

Refresher III

— Inner Product Spaces —

- Inner Products
- Orthogonal Complements, Orthonormal Basis
- Grahm–Schmidt procedure
- Orthogonal Projection

Inner Product Spaces

Definition

An inner product space is a vector space V over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ endowed with an inner product $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{K}$ that satisfies $\forall u, v, w \in V$ and $\lambda \in \mathbb{K}$:

- (symmetry) $\langle v, w \rangle = \overline{\langle w, v \rangle}$
- (linearity) $\langle v + \lambda w, u \rangle = \langle v, u \rangle + \lambda \langle w, u \rangle$
- (definiteness) $\langle v, v \rangle > 0$ if $v \neq 0$.

If $\mathbb{K} = \mathbb{R}$, V is also called an Euclidean space, and if $\mathbb{K} = \mathbb{C}$, V is an unitary space.

Inner Product Spaces — Examples

Example

The standard inner product on $\mathbb{K}^n \simeq \mathbb{K}^{n,1}$ is $\langle v, w \rangle = \bar{w}^T v = \sum_{i=1}^n \bar{v}_i w_i$.

Example

For $M \in \mathbb{K}^{n,n}$ let $M^\dagger := M^H := \bar{M}^T$. Then $\langle M, N \rangle = \operatorname{tr}(N^H M)$.

Example

The L^2 product on the space of smooth functions $[0, 1] \to \mathbb{C}$ is

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

Orthogonality

Definition

- $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$.
- A basis $\mathcal{B} \subset V$ is orthogonal if all pairs $v, w \in \mathcal{B}$, $v \neq w$, are orthogonal.
- This basis is orthonormal if moreover $1 = \|v\| := \sqrt{\langle v, v \rangle}$ for all $v \in \mathcal{B}$.
- The orthogonal complement of a subspace $U \subset V$ is

$$U^{\perp} := \{v \in V \mid \langle u, v \rangle = 0 \; \forall u \in U\}.$$

Orthogonality — Basic Properties

Proposition

- The orthogonal complement $U^\perp$ is always a linear subspace, and $U \cap U^\perp = \{0\}$.
- All pairs $v, w \in V$ satisfy the Cauchy-Schwarz inequality

$$|\langle v, w\rangle| \leq \|v\|\|w\|.$$

- If $v, w \in V$ are orthogonal, then the Pythagorean theorem holds:

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$